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V.A. Marchenko and E.Y. Khruslov: Homogenization of Partial Differential Equations

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This book, which was first published in Russian in 2005, deals with the homogenization of partial differential equations (pdes) of elliptic and parabolic types. The authors study both boundary value problems posed in highly perforated domains or equations with rapidly oscillating coefficients. The standard theory of homogenization (periodic theory, uniformly elliptic oscillating coefficients) is supposed to be already known and the emphasis is put on nonstandard situations leading to multicomponent or nonlocal equations.

Summary of the book The first chapter is an introduction where the considered problems and the main results are presented in an informal manner. The authors make out the different categories of problems which will be treated and supply an example for each of them. It greatly helps the reader finding his way through the book.

Chapters 2 and 3 are devoted to Dirichlet boundary value problems for the stationary Helmholtz equation in perforated domains: Chap. 2 deals with a fixed domain Ω perforated by a large number of disjoint holes with small diameters. Mathematically, one rather considers a sequence of domains Ω^s obtained by removing from Ω a number (tending to infinity as $s \to \infty$) of disjoint closed sets (whose diameters tend to zero). There are different regimes depending on the asymptotic behaviours of the holes' diameters and the free path between the holes. In particular, there are special regimes when a new linear potential term (named by Cioranescu and Murat "the strange term") appears in the homogenized equation. It involves the limiting density of capacity of the holes. The result is proved as a consequence of an abstract theorem on the asymptotics of some sequences of projectors in a Hilbert space. The definition of the so-called correctors is not as simple as in the periodic case. A probabilistic version of the theorem is given as well. Chapter 3 is devoted to connected perforations, e.g. fibers organized in network. Here, a different (variational) method of proof is used.

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Chapters 4 and 5 are devoted to the study of Neumann boundary value problems for the stationary Helmholtz equation in perforated domains. The important notion of families of strongly connected domains is introduced in Chap. 4. Roughly speaking such domains make it possible to construct extension operators uniformly continuous in energy norms. In Chap. 5, the authors make out three categories of domains: the strongly connected domains, the weakly connected domains and the domains with traps. With family of strongly connected domains, the homogenized problem has the classical form of a single pde. For weakly connected domains, homogenization leads to a system of weakly coupled pdes, with as many unknowns as strongly connected components. The authors also consider domains with traps, i.e. there are a large number of pores occupying a macroscopic volume weakly connected to a strongly connected domain. Such situations lead to a homogenized pde where the potential term is a rational function of the zeroth order coefficient of the original pde. Up to my knowledge these results appear for the first time in a monograph.

The parabolic counterparts of the results contained in Chaps. 2–5 are presented in Sect. 6. In particular, it is proven that the previously mentioned domains with traps (Neumann conditions) lead to a parabolic equation with a memory term. The related problem of the convergence of the spectra of the elliptic problems is studied too. Finally, the authors consider Dirichlet problems for the heat equation in perforated domains with evolving holes.

In Chap. 7, the authors investigate parabolic problems with oscillating coefficients but do not suppose that the coefficients are uniformly elliptic or uniformly bounded. For example, they consider the following situation:

- There are a finite number M of disjoint strongly connected regions where the tensor is uniformly elliptic.
- There are other disjoint regions whose diameter tend to zero, where the tensor is uniformly elliptic and which are surrounded by a macroscopic region where the conductivity tensor tends to 0.

In this case, homogenization leads to M coupled parabolic equations with non local in time coupling terms. The well-known double porosity models introduced in geophysics are a particular case of this situation.

Finally, he case of non bounded coefficients is considered. The authors present situations where homogenization leads to nonlocal problems in space. The heat equation for a medium containing inclusions with high heat capacity is studied as well.

Chapter 8 is devoted to the homogenization of problem in domains with small perforations concentrated near a smooth boundary. Homogenization leads to new boundary conditions on the smooth boundary.

Conclusion Most of the results presented in this monograph are nonstandard and cannot be found in the other available books on homogenization. In general, the other existing books are either restricted to periodic media or rather abstract. The present monograph aims at lying in between these two extremes, by tackling concrete but nonperiodic nonstandard cases and yet trying to explicitly compute the homogenized coefficients. Of course, the price for that is an increased technical complexity.

The authors have chosen not to explicitely treat important abstract notions such as G–Γ–H–Mosco convergence and rather refer to other books for that. Similarly, the question of finding bounds on the homogenized coefficients is not discussed.

The book is very well written. Despite their complexity, the results are clearly explained and their proofs are thoroughly written.

A little criticism: the references to the western literature could be more complete and accurate (in particular, almost no American author is cited), and there are many errors in the orthography of the names of the European authors.

To conclude, this is an excellent monograph about advanced results in homogenization. I recommend it to every researcher in this field, although beginners should first read easier books.